Global infrared rearrangements and the renormalisation of QCD

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Based on

JHEP10 (2017) 179, arxiv:1709.08541; To appear

20 Nov 2017 – IIT Hyderabad

Outline

Motivation

Problem: find the UV counterterm $Z(\gamma)$ of a Feynman diagram γ
Applications:

- Compute the singular part of higher-loop diagrams

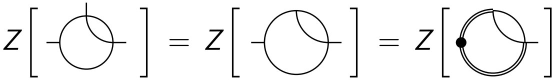
$$K_\epsilon(\Gamma) = -\sum_{\substack{\gamma\in\Gamma \\ \gamma\neq\emptyset}} Z(\gamma) * \Gamma/\gamma \tag{1.1}$$

  K_ϵ extracts the pole part, Γ/γ is the reduced graph.
- Renormalisation group functions (*e.g.* anomalous dimensions)

Origin of IR rearrangements

In MS scheme $Z(\gamma)$ is a polynomial in the masses (Collins, 1977).

- If γ is logarithmically divergent $Z(\gamma)$ is mass-independent.
  γ can always be made log-divergent by taking derivatives.
- *Infrared rearrangements* (IRRs), acting on masses and external momenta, simplify the calculation of $Z(\gamma)$ (Vladimirov, 1980).

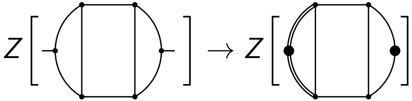


Double lines are massive propagators, dotted lines are squared propagators.

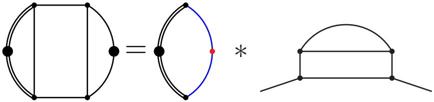
One-mass tadpoles (I)

IRRs reduce the complexity of the calculation of **1 loop**.

- Global IRR applies the same rearrangement to all diagrams.
  Ex: **three-loop** propagator

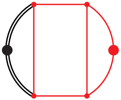


Factorisation into one-loop tadpole and two-loop propagator



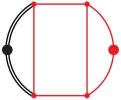
One-mass tadpoles (II)

More in detail



$$= \int d^d q \, \frac{1}{\left[q^2 + M^2\right]^2} * \Pi(q^2)$$

Dimensional analysis: $\Pi(\mathrm{q}^2) = (q^2)^{-2-2\epsilon}\,\zeta$. In conclusion

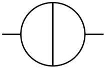
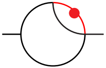


$$= \int d^d q \, \frac{\zeta}{\left[q^2 + M^2\right]^2 (q^2)^{2+2\epsilon}}$$

- IR divergences are generated at $q \to 0$!

Dealing with IR singularities

IRRs can generate IR poles: we need a strategy to deal with them.

 $\longrightarrow$  $\simeq \int \frac{d^d q}{(2\pi)^d} \cdots \int \frac{d^d k}{(2\pi)^d} \frac{1}{(q+k)^2 [k^2]^2}$

- Auxiliary mass regulating the IR (Chetyrkin, Misiak, Munz; van Ritbergen, Larin, Vermaseren '97)
- IR counterterms: apply the R^* operation (Chetyrkin, Smirnov, Tkachov '82, '84-'85)

$$R^*(\Gamma) = \widetilde{R} \circ R(\Gamma) \tag{1.2}$$

$\widetilde{R}$ recursively subtracts IR divergences from each diagram.

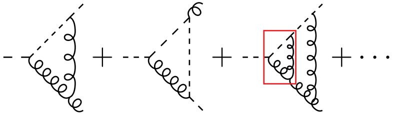
Global approach

The R^* operation is very flexible, because it allows general rearrangements of each Feynman diagram separately.

- IR counterterms must be computed on a diagram-by-diagram basis
- high-order calculation can involve $\mathcal{O}(10^5)$ Feynman diagrams, with many IR counterterms per diagram $\rightarrow$ Bottleneck
- **Global** R^* (Chetyrkin 1991) avoids this problem my using a single IR counterterm for the whole process.

An example

Renormalisation of the ghost-gluon vertex

$\delta\Gamma(g, p) =$ 

The UV vertex divergence is renormalised by Z_1
$g \to g_0$ renormalises all the remaining subdivergences.

$$Z_1 + \delta\Gamma(g_0, p) + \delta Z_1 * \delta\Gamma(g_0, p) = \text{finite} \tag{2.3}$$

An example

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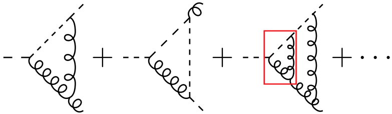

The UV vertex divergence is renormalised by Z_1
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$$\delta Z_1 = -K_\epsilon \left[Z_1 * \delta\Gamma(g_0, p) \right] \tag{2.3}$$

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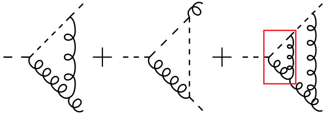


$$+ \cdots$$

The UV vertex divergence is renormalised by Z_1
$g \to g_0$ renormalises all the remaining subdivergences.

$$\boxed{\delta Z_1 = -K_\epsilon\Big[Z_1 * \widetilde{R}\big[\delta\Gamma(p=0)\big]\Big]} \tag{2.3}$$

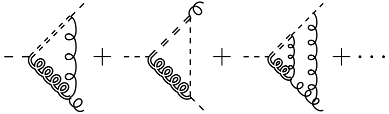
Global R^*

Global infrared rearrangement

We rearrange all the Feynman diagrams into **factorised tadpoles**.

1. Introduction of the mass

$\delta\Gamma_M(g, p, M) =$



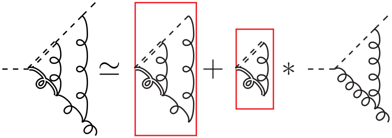
- Vertex divergence (mass independent) renormalises with Z_1
- The **massive vertex** subdivergence changes

$$Z_1 + \delta\Gamma_M(g_0, p, M) + \delta Z_1 * \delta\Gamma(g_0, p) = \text{finite} \qquad (2.4)$$

Tadpole Limit

The Feynman integrals approach tadpoles in the limit $M^2 \gg p$.

2. Hard mass expansion (Smirnov 1996) expands all the subdiagrams involving heavy lines around $p \to 0$



In formula

$$\delta\Gamma_M(g_0, p, M) \simeq \boxed{\delta\Gamma_M(g_0, 0, M)} + \boxed{\delta\Gamma_M(g_0, 0, M)} * \delta\Gamma(g_0, p, 0) \tag{2.5}$$

Global infrared counterterm

Z_1 is almost written as one-mass tadpoles.
$\delta\Gamma(g_0, p)$ has IR poles at $p \to 0$. $\widetilde{R}$ can now subtract them at global level.

$$\delta Z_1 = -K_\epsilon \left[\underbrace{\delta\Gamma_M(g_0, 0, M)}_{\text{tadpole}} + \left(\delta\Gamma_M(g_0, 0, M) + \delta Z_1\right) * \delta\Gamma(g_0, p) \right] \tag{2.6}$$

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$\widetilde{R}\Big[\delta\Gamma(p = 0)\Big]$ is imposed by multiplicative renormalisation

$$\boxed{\delta Z_1 = -Z_1 * \widetilde{R}\Big[\delta\Gamma(g_0, p = 0)\Big]} \tag{2.7}$$

Global R^* at work

1. Renormalisation of the globally rearranged diagrams gives

$$\delta Z_1 = -K_\epsilon \left[\delta\Gamma_M(0, M) + \Big(\delta\Gamma_M(0, M) + \delta Z_1 \Big) * \widetilde{R}\Big[\delta\Gamma(p = 0) \Big] \right]$$

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2. introduction of the IR counterterm $\widetilde{R}\left[\delta\Gamma(p = 0)\right]$

$$\widetilde{R}\left[\delta\Gamma(p = 0)\right] = -\frac{\delta Z_1}{Z_1}.$$

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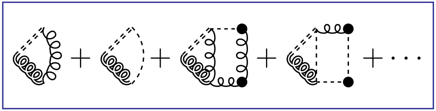
$$\widetilde{R}\Big[\delta\Gamma(p=0)\Big] = -\frac{\delta Z_1}{Z_1}.$$

3. In conclusion Z_1 is determined

$$\delta Z_1 = -K_\epsilon\left[\frac{\delta\Gamma_M(g_0,0,M)}{Z_1} - \frac{(\delta Z_1)^2}{Z_1}\right] \qquad (2.8)$$

Towards 5-loop renormalisation

$$\delta Z_1 = -K_\epsilon \left[\frac{\boxed{\delta\Gamma_M(g_0, 0, M)}}{Z_1} - \boxed{\frac{(\delta Z_1)^2}{Z_1}} \right] \tag{3.9}$$



Counterterms from lower orders.

- We reduced the calculations to factorized L-loop tadpoles.
- We automated the computation of factorized tadpoles using `FORCER` (Ruijl, Ueda, Vermaseren 2017) and determined Z_1 **to 5 loops.**

Highly efficient approach

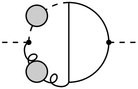
Renormalisation of QCD

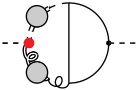
- The ghost-gluon vertex renormalisation constant Z_1 is the first necessary ingredient to renormalise QCD.
- One of the possible choices includes
  - The ghost field renormalisation Z_3^c
  - The gluon field renormalisation Z_3
  - The quark field renormalisation Z_2
- Ward identities fix the remaining constants

$$Z_g = \frac{Z_1}{Z_3^c\sqrt{Z_3}} = \frac{Z_1^{qqg}}{Z_2\sqrt{Z_3}} = \frac{Z_1^{3g}}{(Z_3)^{\frac{3}{2}}} \tag{3.10}$$

The ghost field renormalisation

- Let's consider now the global rearrangement for the ghost field

$\Pi(q^2) =$  $\rightarrow K_\epsilon\left[Z_3^c\left(1+\Pi(g_0,q^2)\right)\right] = 0$

$\Pi(q^2) =$  $\rightarrow K_\epsilon\left[Z_3^c\left(1+\frac{\Pi_M(q^2,M^2)+\delta Z_1\cdot\Pi(q^2)}{Z_1}\right)\right] = 0$

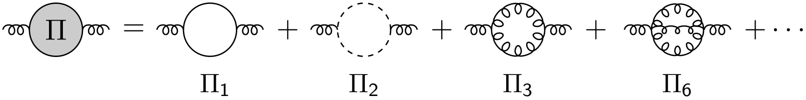
- Using the hard mass expansion and the IR counterterm

$$\delta Z_3^c = -K_\epsilon\left[\frac{Z_3^c}{Z_1}\left(\Pi_M(0,M) - \frac{\delta Z_3^c}{Z_3^c}\left(\delta\Gamma_M(0,M)+\delta Z_1\right)\right)\right].$$

The gluon field renormalisation

- The gluon renormalisation Z_3 is complicate: first time undertaken in 2016 for $SU(3)$ (Baikov, Chetyrkin, Kühn).

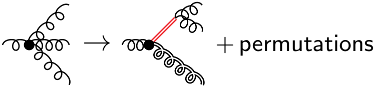
$$K_\epsilon\left[Z_3\left(1+\sum_{i=1,2,3,6}\Pi_i(g_0,q)\right)\right]=0, \qquad (3.11)$$



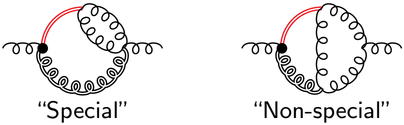
(a) New rearrangements of 4-point vertices.
(b) External gluons have many interactions.

Some problems (a)

- Mass insertion in the 4-gluon vertex via an auxiliary field



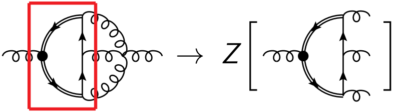
- New "special", self-energy-like, subdivergences



Some problems (b)

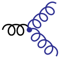
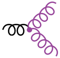
Highly non-trivial renormalisation of the modified vertices:

- vertices mix among each other. *E.g.*



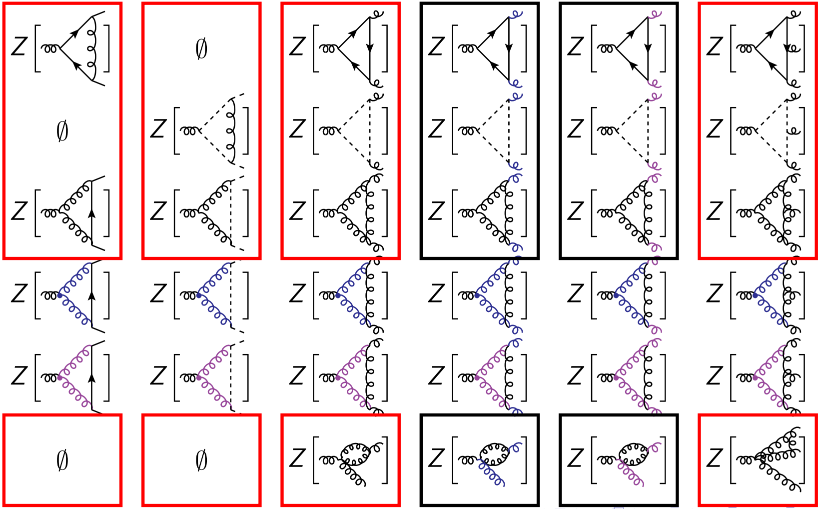
counterterm mixing quark and 4-gluon operators.

- Mixing into new operators (not in the QCD Lagrangian).
  We analysed the operators appearing for general gauge group
  - Two new 3-gluon vertices

$O_4 =$  $O_5 =$ 

  - New 4-gluon vertices appear at each loop order: we selected a basis of six new vertices $O_7, .., O_{12}$ needed to this order. Note that each of them is splitted with the auxiliary field.

A 12x12 mixing matrix



Solution

Only a small subset of the matrix elements z_{ij} contributes and

$$\sum_{i=1,2,3,6} z_{ij} = \begin{cases} 0, & \text{if } j \in \{4,5,7,..12\} \text{ is a gauge variant operator} \\ Z_1^j, & \text{if } j \in \{1,2,3,6\} \text{ is a QCD operator} \end{cases}$$

where $Z_1^j \in \{Z_1^{ccg}, Z_1^{qqg}, Z_1^{ggg}, Z_1^{gggg}\}$ are QCD vertex renormalisation constant.

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Exploiting these features we write the gluon renormalisation constant in terms of factorized tadpoles $\Pi_i(M)$ and lower order counterterms

$$\begin{aligned} \delta Z_3 &= -K_\epsilon \Bigg\{ \sqrt{Z_3} \sum_{i=1,2,3,6} \Bigg\{ \sum_{j,k} \Big[z_{ij}^{sp} \mathbf{Z_j} \Big(\Pi_j(M) + \delta\Gamma_{jk}^{ns} \Pi_k(q) \Big) \Big] \\ &+ (\sqrt{Z_3} - \mathbf{Z}_i) \Pi_i(q) - \sum_j \delta z_{ij}^{sp} \mathbf{Z}_j \Pi_j(q) \Bigg\} \Bigg\}. \end{aligned}$$

Results

- Z_1^{ccg}, Z_1^{qqg}, Z_2, Z_3^c to 5 loops with all the powers of the gauge parameter ξ.
  - Verified $Z_1^{ccg} \propto (1-\xi)$.
  - Verified consistency with the Ward identities.
- Z_3 to 5 loops, retaining linear terms in ξ.
- We computed the coupling renormalisation

$$Z_\alpha = \frac{(Z_1^{ccg})^2}{Z_3\,(Z_3^c)^2}, \tag{4.12}$$

  - Verified independence on ξ to first order.

Complete renormalisation of QCD to 5 loops in covariant gauges for general gauge group.

Landau gauge quark anomalous dimension

$$
\begin{aligned}
(\gamma_2)_4 \quad &= C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[-\frac{1985}{24} + \frac{781753}{192} \zeta_7 - \frac{1458845}{384} \zeta_5 + \frac{135731}{192} \zeta_3 + \frac{3577}{64} \zeta_3^2 \right] \\
&+ T_R n_f \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{6200}{9} - \frac{1425}{4} \zeta_6 + \frac{27377}{6} \zeta_7 + \frac{1113}{4} \zeta_4 - \frac{9915}{2} \zeta_5 \right. \\
&\left. - \frac{2468}{3} \zeta_3 + \frac{91}{2} \zeta_3^2 \right] + T_R n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{7360}{9} + 640 \zeta_5 + \frac{704}{3} \zeta_3 \right] \\
&+ C_F n_f^4 T_R^4 \left[\frac{1328}{243} - \frac{256}{27} \zeta_3 \right] + C_F \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{113}{6} - \frac{125447}{8} \zeta_7 \right. \\
&\left. + 1015 \zeta_5 + 17554 \zeta_3 - 4884 \zeta_3^2 \right] + C_F n_f \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{5984}{3} - 8680 \zeta_7 \right. \\
&\left. + 18080 \zeta_5 - 12096 \zeta_3 + 3648 \zeta_3^2 \right] + C_F^2 n_f^3 T_R^3 \left[-\frac{2636}{243} - 64 \zeta_4 + \frac{832}{9} \zeta_3 \right] \\
&+ C_F^3 n_f^2 T_R^2 \left[-\frac{2497}{27} - 128 \zeta_4 + 320 \zeta_5 + \frac{400}{9} \zeta_3 \right] + C_F^4 n_f T_R \left[\frac{29209}{36} \right. \\
&\left. + \frac{6400}{3} \zeta_6 - 800 \zeta_4 - \frac{46880}{9} \zeta_5 + \frac{22496}{9} \zeta_3 + \frac{1024}{3} \zeta_3^2 \right] + C_F^5 \left[\frac{4977}{8} \right. \\
&\left. - 47628 \zeta_7 + 22600 \zeta_5 + 16000 \zeta_3 + 2496 \zeta_3^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
&+C_A \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[-\frac{173959}{144} + \frac{15675}{16}\zeta_6 + \frac{3016307}{256}\zeta_7 - \frac{12243}{16}\zeta_4 \right. \\
&\left. + \frac{609425}{96}\zeta_5 - \frac{574393}{32}\zeta_3 + \frac{16935}{4}\zeta_3^2 \right] \\
&+C_A n_f \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[\frac{33464}{9} + \frac{23632}{3}\zeta_7 - \frac{48640}{3}\zeta_5 + 8992\,\zeta_3 - 2320\,\zeta_3^2 \right] \\
&+C_A C_F n_f^3 T_R^3 \left[-\frac{3566}{243} + 64\,\zeta_4 - \frac{1984}{27}\zeta_3 \right] + C_A C_F^2 n_f^2 T_R^2 \left[\frac{101485}{162} \right. \\
&\left. + \frac{1600}{3}\zeta_6 + 176\,\zeta_4 - \frac{3712}{3}\zeta_5 - \frac{6160}{9}\zeta_3 + \frac{256}{3}\zeta_3^2 \right] + C_A C_F^3 n_f T_R \left[-\frac{167263}{108} \right. \\
&\left. -4800\,\zeta_6 - 13944\,\zeta_7 + 2120\,\zeta_4 + \frac{58720}{3}\zeta_5 - \frac{25804}{9}\zeta_3 - 64\,\zeta_3^2 \right] \\
&+C_A C_F^4 \left[-\frac{835739}{144} - \frac{17600}{3}\zeta_6 + 123977\,\zeta_7 + 2200\,\zeta_4 - \frac{248960}{9}\zeta_5 - \frac{530884}{9}\zeta_3 \right. \\
&\left. -\frac{24632}{3}\zeta_3^2 \right] + C_A^2 C_F n_f^2 T_R^2 \left[\frac{120037}{162} - \frac{800}{3}\zeta_6 - \frac{441}{2}\zeta_7 - 179\,\zeta_4 + \frac{3584}{9}\zeta_5 \right. \\
&\left. + \frac{3140}{3}\zeta_3 - \frac{128}{3}\zeta_3^2 \right] + C_A^2 C_F^2 n_f T_R \left[\frac{717409}{432} + 1150\,\zeta_6 + \frac{42203}{3}\zeta_7 - \frac{1411}{4}\zeta_4 \right. \\
&\left. -\frac{95792}{9}\zeta_5 - \frac{14287}{24}\zeta_3 - 1214\,\zeta_3^2 \right] + C_A^2 C_F^3 \left[\frac{827215}{72} + 13200\,\zeta_6 - \frac{1789067}{16}\zeta_7 \right. \\
&\left. -4664\,\zeta_4 - \frac{188795}{12}\zeta_5 + \frac{1365227}{18}\zeta_3 + \frac{18097}{2}\zeta_3^2 \right] + C_A^3 C_F n_f T_R \left[-\frac{31919039}{7776} \right. \\
&\left. + \frac{4825}{16}\zeta_6 - \frac{440419}{144}\zeta_7 - \frac{8705}{32}\zeta_4 + \frac{28721}{18}\zeta_5 - \frac{144377}{864}\zeta_3 + \frac{4067}{6}\zeta_3^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
&+C_A^3\,C_F^2\left[-\frac{42214139}{3888}-\frac{43175}{6}\,\zeta_6+\frac{9074513}{192}\,\zeta_7+\frac{3815}{4}\,\zeta_4+\frac{5957573}{288}\,\zeta_5\right.\\
&\left.-\frac{5503507}{144}\,\zeta_3-\frac{78041}{24}\,\zeta_3^2\right]+C_A^4\,C_F\left[\frac{368712343}{62208}+\frac{227975}{192}\,\zeta_6\right.\\
&\left.-\frac{312820991}{36864}\,\zeta_7+\frac{87067}{128}\,\zeta_4-\frac{16237513}{3072}\,\zeta_5+\frac{46196783}{6912}\,\zeta_3+\frac{23555}{128}\,\zeta_3^2\right].
\end{aligned}
$$

- More complicated structure compared to the beta-function:
  - zeta-values up to weight 7.
- The leading and next-to-leading n_f terms agree with the all-order results of (Ciuchini et al. '99).

Outlook

- The global R^* method is highly efficient from the computational point of view.
- On the other hand, it is process-dependent $\rightarrow$ difficult to generalize and automate.
- We tackled the rearrangement of different operators $\rightarrow$ case study for more applications:
- Determination of the moments of the $N4LO$ splitting functions, whose calculation was shown to be highly demanding.

Thank you